

Fractional Quantum Hall Effects in Graphene and Its Bilayer

Naokazu Shibata and Kentaro Nomura¹

¹Department of Physics, Tohoku University, Aoba, Aoba-ku, Sendai, 980-8578, Japan

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Single-layer and Bilayer of graphene are new classes of two-dimensional electron systems with unconventional band structures and valley degrees of freedom. The ground states and excitations in the integer and fractional quantum Hall regimes are investigated on torus and spherical geometries with the use of the density matrix renormalization group (DMRG) method. At nonzero Landau level indices, the ground states at effective filling factors 1, 1/3, 2/3 and 2/5 are valley polarized both in single-layer and bilayer graphenes. We examine the elementary charge excitations which could couple with the valley degrees of freedom (so called valley skyrmions). The excitation gaps are calculated and extrapolated to the thermodynamic limit. The largest excitation gap at effective filling 1/3 is obtained in bilayer graphene, which is a good candidate for experimental observation of fractional quantum Hall effect.

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I. INTRODUCTION

A recent experimental realization of single-layer graphene (SLG) sheets¹ has made it possible to confirm a number of theoretical predictions of intriguing electric properties of massless Dirac fermion systems,² including unconventional quantum Hall effects (QHE)^{3,4} with the half-integer Hall conductivity⁵

$$\sigma_{xy} = (n + \frac{1}{2}) \frac{4e^2}{h}, \quad (1)$$

at $\nu = \pm 2, \pm 6, \pm 10, \dots$, where a factor 4 is the Landau level (LL) degeneracy, resulting from spin and valley (referred to K and K') symmetry in graphene. $\nu = 2\pi\ell_B^2\rho$ is the filling factor, $\ell_B = \sqrt{\hbar/eB}$ is the magnetic length, ρ is the carrier density measured from the charge neutral Dirac point.

As many phenomenological insights were found in the quantum Hall regime of SLG^{3,4,5,6,7,8,9,10,11,12,13,14,15}, bilayer graphene (BLG) potentially exhibits rich physics.^{16,17} Experiments showed that the Hall conductivity of unbiased graphene bilayer in strong magnetic fields is given by

$$\sigma_{xy} = n \frac{4e^2}{h} \quad (2)$$

with $|n| \geq 1$.¹⁶ The absence of $\sigma_{xy} = 0$ plateau and the double height jump of the Hall conductivity between $\nu = -4$ and 4 indicate eight-fold degeneracy at the neutrality point.

These quantization rules of the Hall conductivity for SLG and BLG originate with the characteristic energy spectrum in a magnetic field. The low-energy band structure in SLG consists of Dirac cones located at the inequivalent Brillouin zone corners K and K'. Using the magnetic ladder operator $a \equiv (\pi_x - i\pi_y)\ell_B/\sqrt{2\hbar}$, where $\pi = \mathbf{p} - e\mathbf{A}$ is the kinetic momentum, the single-particle

Hamiltonian for the K-valley of the SLG is written as

$$\mathcal{H}_K^{(\text{SLG})} = \frac{\sqrt{2}\hbar v_F}{\ell_B} \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix}, \quad (3)$$

and $\mathcal{H}_{K'}^{(\text{SLG})}$ is the transpose of $\mathcal{H}_K^{(\text{SLG})}$. Where v_F is the Fermi velocity of SLG, 2×2 matrices act in the sublattice degrees of freedom in graphene. The eigenenergy of eq.(3) is given by

$$\epsilon_n = \pm \hbar v_F \sqrt{2n}/\ell_B. \quad (4)$$

The eigenvector is $|0\rangle_K^{(\text{SLG})} = (0, |0\rangle)^t$ for $n = 0$, and $|n, \pm\rangle_K^{(\text{SLG})} = (|n-1\rangle, \pm|n\rangle)^t/\sqrt{2}$ for $n \geq 1$. Here $|n\rangle$ is the eigenvector of the number operator $N \equiv a^\dagger a$ with an eigenvalue n .

In contrast to SLG, BLG has an ordinary parabolic spectrum in the vicinity of the neutrality point. In a magnetic field, the effective Hamiltonian for BLG is written in the form:¹⁷

$$\mathcal{H}_K^{(\text{BLG})} = \hbar\omega_c \begin{pmatrix} 0 & a^2 \\ (a^\dagger)^2 & 0 \end{pmatrix}, \quad (5)$$

and its eigenenergy is

$$E_n = \pm \hbar\omega_c \sqrt{n(n-1)}. \quad (6)$$

Here $\omega_c = eB/m$ with m being the effective mass of BLG. The eigenvector of eq.(5) is $|0, 0\rangle_K^{(\text{BLG})} = (0, |0\rangle)^t$, $|0, 1\rangle_K^{(\text{BLG})} = (0, |1\rangle)^t$ for the zero energy level, and $|n, \pm\rangle_K^{(\text{BLG})} = (|n-2\rangle, \pm|n\rangle)^t/\sqrt{2}$ for $n \geq 2$.

Recent experimental studies on SLG in a sufficiently strong magnetic field revealed new quantum Hall states at $\nu = 0, \pm 1, \pm 4$,⁶ where the electron-electron interaction may play a crucial role. Here relevant energy scales in graphene in a magnetic field are

(i) LL separation around the neutrality point, $\sqrt{2}\hbar v_F/\ell_B \simeq 400\sqrt{B[T]}[K]$ for SLG, while $\sqrt{2}\hbar\omega_c \simeq 30 \times (B[T])[K]$ for BLG.

- (ii) Zeeman coupling, $\Delta_z \equiv g\mu_B|\mathbf{B}| \simeq 1.5 \times (B[T])[K]$, and
 (iii) The Coulomb energy, $e^2/\epsilon\ell_B \simeq 100\sqrt{B[T]}[K]$.

The activation energy measurements at these additional QHE states in SLG⁶ have shown that at $\nu = \pm 4$ the gap has linear B dependence and reasonably corresponds to Δ_z , indicating Zeeman spin splitting. At $\nu = \pm 1$, on the other hand, the gap is approximately scaled by \sqrt{B} , that indicates the gap originates from the Coulomb interaction. The latter behavior is consistent with the quantum Hall ferromagnetism (QHF)¹⁸ in which valley degrees of freedom, referred as pseudospins, spontaneously split via the exchange energy at *all integer* fillings.^{7,8,9,10,11,12,13} Based on the above scenario, it has been anticipated that interactions will drive quantum Hall effects also in BLG, at the octet's seven intermediate integer filling factors when magnetic field is strong enough or disorder is weak enough.¹⁹

In this work, we focus on the many-body states in nonzero LLs of SLG and BLG. We introduce the effective filling factors of the topmost partially filled n th LL,

$$\nu_n^{(\text{SLG})} = \nu - 4(n - 1/2) \quad (7)$$

for SLG and

$$\nu_n^{(\text{BLG})} = \nu - 4(n - 1) \quad (8)$$

for BLG. Here $\nu_n \leq 4$, and $n \geq 2$ for BLG.

The ground state at $\nu_n = 1$ is fully spin and valley polarized, and the wave function can be represented by

$$|\Psi_{\tau}^{\nu_n=1}\rangle = \prod_m c_{m,\tau}^{\dagger} |0\rangle, \quad (9)$$

where we assign the valley K and K' in graphene as z -component of pseudospin $\tau = K$ or K' . We omit real spins, which are assumed to be fully polarized by the strong Zeeman splitting. Excitations from the symmetry broken states are described by (pseudo)spin wave and (pseudo)spin textures called skyrmions^{20,21,22,23,24} or other types, depending on the LL index n .⁸

In the previous work, we have studied the fractional quantum Hall ferromagnetic states in SLG at $\nu_n = 1$ and $1/3$.¹⁵ As a consequence of the relativistic nature of electrons in SLG, the effective electron-electron interactions in $n \neq 0$ LLs differ from that in conventional two-dimensional systems, while the $n = 0$ LL is equivalent.^{7,10,14} We have neglected the spin degrees of freedom of electrons by assuming strong Zeeman coupling, although valley degrees of freedom have been taken into account since there is no external symmetry breaking field. We have shown that the ground states at $\nu_n = 1$ and $1/3$ in $n = 0$ and 1 LLs of SLG are fully valley polarized, while elementary charge excitations consist of pseudospin-singlet, namely valley-skyrmions.^{20,21,22,23,24}

In this paper we extend the quantum Hall ferromagnetism to BLG and numerically show that the ground states at $\nu_n = 1$ and $1/3$ in the $n = 2$ LL of BLG are

fully valley polarized. In contrast to the fact that elementary charge excited states are pseudospin-singlet at $\nu_2 = 1$, those at $\nu_2 = 1/3$ of BLG are pseudospin polarized. Namely, the Laughlin type quasiparticle (quasihole) excitations dominate over skyrmion type. As a consequence, the charge gap at $\nu_2 = 1/3$ in BLG is almost twice larger than that at $\nu_0 = 1/3$ in SLG. Therefore clean BLG samples are better candidates to observe fractional QHE than SLG. We also study $\nu_n = 2/3$ and $2/5$ states in SLG and BLG. We claim that the ground at $\nu_n = 2/3$ and $2/5$ states are pseudospin singlet in the $n = 0$ LL of SLG, while fully pseudospin polarized in the $n = 1$ LL of SLG and the $n = 2$ LL of BLG. These results are summarized in Table I.

II. MODEL AND METHOD

We start with the projected Coulomb interaction Hamiltonian onto a certain LL, and study charge and valley excitations, where we treat the valley degrees of freedom K and K' in the language of the pseudospin, while real spin degrees of freedom are supposed to be frozen by the Zeeman splitting. The LL mixing is also neglected in the following. We calculate the exact wave function of the ground state and low energy excited states in SLG and BLG, basing on the density matrix renormalization group (DMRG) method,^{25,26,27} and examine the existence of the skyrmion excitations in valley degrees of freedom not only at integer fillings $\nu_n = 1$ but also at fractional fillings $\nu_n = 1/3$, $2/5$ and $2/3$ for the LL indices $n = 0$, 1 and 2 . Charge gaps at these fractions are extrapolated to the thermodynamic limit. For our purpose the DMRG method is quite useful, since it needs to treat a large number of basis of the many-body Hilbert space when the pseudospin degrees of freedom are introduced in the fractional QHE systems. Taking account of valley degrees of freedom, we apply the DMRG method²⁵ on torus^{26,27} and spherical geometries.^{15,28,29,30,31,32}

The projected Hamiltonian onto the n th LL is written as²⁹

$$H = \frac{1}{L^2} \sum_{i < j} \sum_{\mathbf{q}} V(q) e^{-q^2/2} [F_n(q)]^2 e^{i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)}, \quad (10)$$

where \mathbf{R}_i is the guiding center coordinate of the i th particle. The *relativistic* form factor in the n th LL of SLG is written as⁷

$$F_0^{(\text{SLG})}(q) = L_0 (q^2/2) \quad (11)$$

and

$$F_{n \geq 1}^{(\text{SLG})}(q) = (1/2) [L_{|n|} (q^2/2) + L_{|n|-1} (q^2/2)]. \quad (12)$$

For the $n (\geq 2)$ th LL in BLG,

$$F_{n \geq 2}^{(\text{BLG})}(q) = (1/2) [L_{|n|} (q^2/2) + L_{|n|-2} (q^2/2)]. \quad (13)$$

Here $L_n(x)$ are the Laguerre polynomials. In the spherical geometry, it is convenient to write the Hamiltonian as

$$H^{(n)} = \sum_{i < j} \sum_m V_m^{(n)} P_{ij}[m], \quad (14)$$

where $P_{i,j}[m]$ projects onto states in which particles i and j have relative angular momentum $\hbar m$, and $V_m^{(n)}$ is their interaction energy in the n th LL.²⁹ Using above form factors, the pseudopotentials^{28,29} are given by

$$V_m^{(n)} = \int_0^\infty \frac{dq}{2\pi} q V(q) e^{-q^2} [F_n(q)]^2 L_m(q^2). \quad (15)$$

The corresponding integrals for electrons on the surface of a sphere which are used in the present work are described in refs. 29,30,31. Note that our Hamiltonian eqs. (10) and (14) have SU(2) symmetry in the valley degrees of freedom. A symmetry breaking correction to eqs. (10) and (14) which stems from the lattice structure of SLG^{9,10} and BLG is order of a/ℓ_B (a being a lattice spacing) in units of e^2/ℓ_B and neglected in the following.

We calculate the ground state wave function using the DMRG method,^{25,26} which is a real space renormalization group method combined with the exact diagonalization method. The DMRG method provides low-energy eigenvalues and corresponding eigenvectors of the Hamiltonian within a restricted number of basis states. The accuracy of the results is systematically controlled by the truncation error, which is smaller than 10^{-4} in the present calculation. We investigate systems of various sizes with up to 40 electrons in the unit cell keeping 1400 basis in each block.^{26,27}

The sphere geometry is useful to extrapolate energy gaps to the thermodynamic limit. In the sphere geometry, the pseudospin (valley) polarized ground state at $\nu_n = 1/q$ (q being an odd integer), the Laughlin state,²⁸ realizes when the total flux N_ϕ is given by²⁹

$$N_\phi(\nu_n, N_e) = \nu_n^{-1}(N_e - 1), \quad (16)$$

where N_e is the number of electrons in the system. Elementary charged excitations from this pseudospin polarized ground state correspond to the ground state configurations of the system with additional/missing flux ± 1 . At $\nu_n = 1/q$, we study two types of excitations: Laughlin's quasiholes (quasiparticles)²⁸ and skyrmion quasiholes (quasiparticles).^{20,21,22,23,24} Laughlin's quasiholes (quasiparticles) correspond to pseudospin polarized excitations with ± 1 flux, whose creation energy is given by

$$\Delta_c^\pm = E(N_\phi \pm 1, P = 1) - E(N_\phi, P = 1), \quad (17)$$

where \pm represents quasiholes and quasiparticles, respectively, and P is the polarization ratio of the pseudospin, i.e. $P \equiv (N_K - N_{K'})/(N_K + N_{K'})$ with N_K ($N_{K'}$) being the number of electrons in K (K') valley.

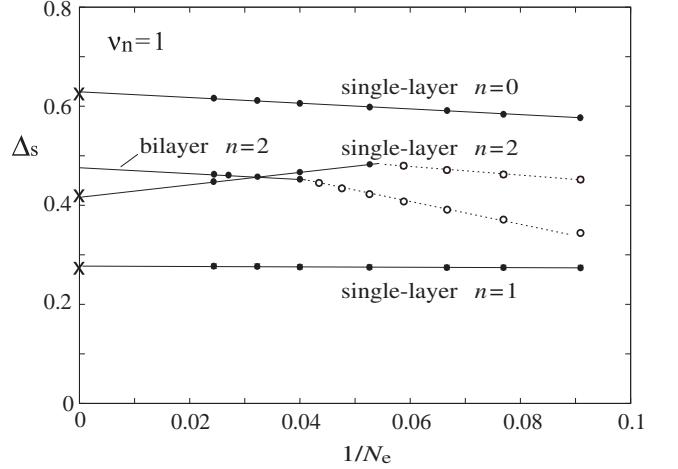


FIG. 1: The lowest charge excitation gaps in SLG and BLG at $\nu_n = 1$ in the $n = 0, 1$ and 2 LLs in the spherical geometry. Closed circles represent the pseudospin unpolarized $P = 0$ (skyrmion) excitation gaps and open circles represent nearly pseudospin-polarized $P = 1 - 2/N_e$ excitation gaps. The crosses on the vertical axis represent results obtained by Hartree-Fock calculations.⁸

Skyrmion quasiholes (quasiparticles) correspond to pseudospin singlet excitations, and their creation energy is given by

$$\Delta_s^\pm = E(N_\phi \pm 1, P = 0) - E(N_\phi, P = 1), \quad (18)$$

which could be smaller than Δ_c^\pm .^{20,21,22,23,24}

The activation energy, referred as the gap in the following, is given as a sum of these quasiholes and quasiparticle energies, $\Delta_c = \Delta_c^+ + \Delta_c^-$ for pseudospin polarized (Laughlin-like) excitations, and $\Delta_s = \Delta_s^+ + \Delta_s^-$ for pseudospin unpolarized (skyrmion-like) excitations.

At $\nu_n = 2/5$ and $2/3$, the pseudospin-polarized ground state and unpolarized ground state compete each other. Unfavorably, in the spherical geometry, these ground states realize at different configurations (the number of electrons and the total flux). Precisely, the pseudospin-polarized $\nu_n = 2/5$ state with N_e electrons occurs when the total flux N_ϕ is given by $N_\phi = (5/2)N_e - 4$. On the other hand, the pseudospin-singlet (unpolarized) state is realized at $N_\phi = (5/2)N_e - 3$. Since the finite size effects are different between these different configurations, it is difficult to study the pseudospin-polarizability in the ground state in the sphere geometry. To avoid this difficulty, we utilize the torus geometry in which two ground states with $P = 0$ and $P = 1$ are realized in the same configuration: $N_\phi = (5/2)N_e$. We use the sphere geometry to extrapolate the energy gaps to the thermodynamic limit. The $\nu_n = 2/3$ states are studied as well in the following section. The elementally charge excitations at $\nu_n = 2/3$ are obtained by adding or removing single electron in the system with ± 1 flux.

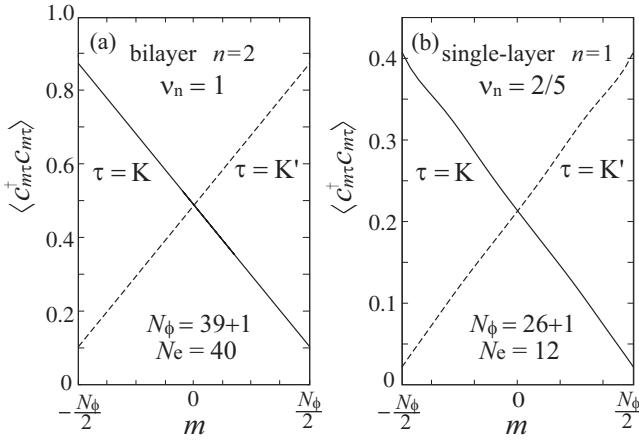


FIG. 2: Expectation values, $\langle c_{m\tau}^\dagger c_{m\tau} \rangle$, in the lowest pseudospin (valley) unpolarized excited state with extra one flux at (a) $\nu_2 = 1$ in BLG and at (b) $\nu_1 = 2/5$ in SLG.

III. RESULTS

A. $\nu_n = 1$ and $1/3$ states

At $\nu_n = 1$ integer fillings, the pseudospins of valley degrees of freedom in SLG and BLG are completely polarized by the exchange Coulomb interaction. This is essentially the same as the case of usual quantum Hall systems with spin degrees of freedom. The fully pseudospin polarized $P = 1$ quasiparticle excitations from the above ground state need energy to the next higher LL, and such excitations have large gap compared with the unpolarized $P = 0$ or partially polarized $0 < P < 1$ excitations in the same LL. We therefore calculate the unpolarized and partially polarized excitations by the DMRG method, and study the pseudospin polarization of the lowest charge excitation and the gap in the thermodynamic limit.

The pseudospin polarization and the energy of the lowest charge excitation are shown in Fig. 1 for various sizes of system in spherical geometry. This figure shows that the pseudospin unpolarized ($P = 0$) state is the lowest excited state when the system size is large enough in SLG ($n = 0, 1$, and 2) and BLG ($n = 2$). The extrapolated value of the pseudospin unpolarized excitation gaps Δ_s in units of $e^2/(\epsilon\ell_B)$ are $0.63, 0.28$ and 0.42 in the $n = 0, 1$ and 2 LLs of SLG, and 0.48 in the $n = 2$ LL of BLG.

In the $n = 2$ LL, the lowest quasiparticle excitation in small systems ($N_e \lesssim 20$) has large pseudospin polarization $P = 1 - 2/N_e$ both for SLG and BLG. These results show instability of the unpolarized $P = 0$ excitations in small systems. We find the first order transition in the lowest charge excited state when the number of electrons N_e exceeds 18 in SLG and 24 in BLG. This is consistent with the expectation that the unpolarized excitations are unstable in higher LLs because of the long-range nature of the effective exchange interaction, which increases energy of skyrmion-like pseudospin unpolarized state.

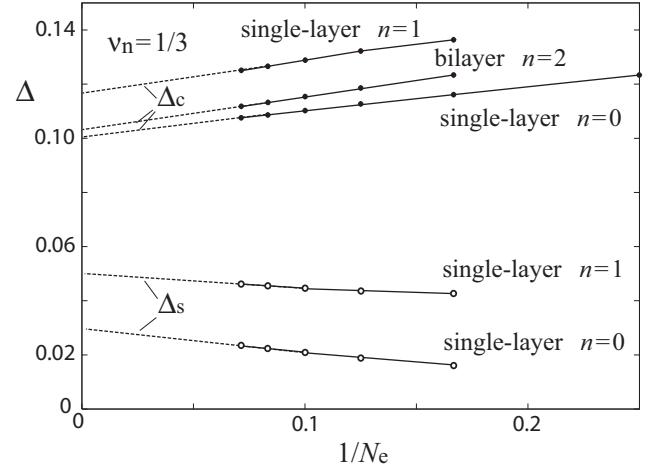


FIG. 3: The pseudospin (valley) polarized excitation gap Δ_c and the pseudospin (valley) unpolarized (skyrmion) excitation gap Δ_s at $\nu_n = 1/3$ in SLG ($n = 0$ and 1 LLs) and in BLG ($n = 2$ LL).

To study the pseudospin structure in the unpolarized excited states, we compare our numerical results with the Hartree-Fock (HF) trial states of skyrmions. The HF trial state of quasi-hole skyrmions at $\nu_n = 1$ is written in the form:¹⁸

$$|\Psi_{sk}\rangle = \prod_{m=-N_\phi/2}^{N_\phi/2} [\alpha_m c_{mK}^\dagger + \beta_m c_{m+1K'}^\dagger] |0\rangle, \quad (19)$$

where $\langle c_{mK}^\dagger c_{mK} \rangle = |\alpha_m|^2$ and $\langle c_{mK'}^\dagger c_{mK'} \rangle = |\beta_{m-1}|^2$. we have calculated the expectation values $\langle c_{m\tau}^\dagger c_{m\tau} \rangle$ from the wave function obtained in the present DMRG study. The results for $\nu_2 = 1$ in BLG indicate they are approximately given by $\langle c_{m\tau}^\dagger c_{m\tau} \rangle = 1/2 \mp m/N_\phi$ for $\tau = K$ and $\tau = K'$, respectively as shown in Fig. 2 (a). Similar results are also obtained for $\nu_n = 1$ and $1/3$ in the $n = 0$ and 1 LLs of SLG, that shows the pseudospin unpolarized elementally charge excitations in BLG at $\nu_2 = 1$ are pseudospin (valley) skyrmions.

The elementally charge excitation gaps at fractional fillings $\nu_n = 1/3$ are shown in Fig.3. The pseudospin (valley) polarized $P = 1$ excitation gap Δ_c for $n = 0$ SLG is 0.101 in units of $e^2/(\epsilon\ell_B)$ in the thermodynamic limit, which is in a good agreement with the previous work.³⁰ In the $n = 1$ LL of SLG, Δ_c is 0.115 , which is larger than 0.101 in the $n = 0$ LL of SLG. In BLG, $\Delta_c = 0.103$ in the $n = 2$ LL, which is also slightly larger than that in the $n = 0$ LL of SLG. This enhancement of the Δ_c in higher LLs is consistent with the increase of the difference of Haldane's pseudopotentials between $m = 1$ and 3 ; $V_1^n - V_3^n$ is 0.168 in units of $e^2/(\epsilon\ell_B)$ for BLG in the $n = 2$ LL and 0.198 for SLG in the $n = 1$ LL, while it is 0.166 for SLG in the $n = 0$ LL.

As seen in conventional 2DEGs, $\nu = 1/3$ (valley or spin) unpolarized excited states can be lower in energy

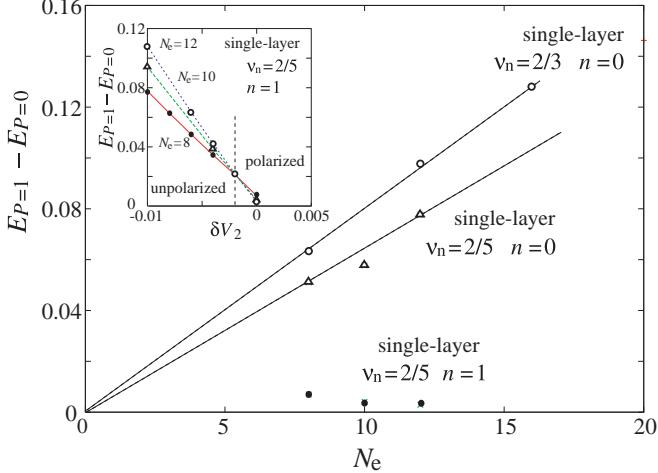


FIG. 4: Size dependence of the energy difference between the fully pseudospin polarized $P = 1$ state $E_{P=1}$ and the unpolarized $P = 0$ states $E_{P=0}$ at $\nu_n = 2/5$ in SLG in a torus geometry. Inset shows size dependence and V_2 dependence of $E_{P=1} - E_{P=0}$ at $\nu_n = 2/5$ in SLG with $n = 1$.

than the polarized excited states. Indeed, as shown in Fig. 3, the unpolarized excitation gap Δ_s in SLG is 0.05 in the $n = 0$ LL and 0.03 in the $n = 1$ LL, which are much smaller than the polarized excitation gap Δ_c in the $n = 0$ and 1 LLs, respectively. In the case of BLG, however, we find that valley unpolarized excited states have higher energy than polarized excited states. Consequently, the activation energy at $\nu_2 = 1/3$ in BLG is 2-3 times larger than that in SLG, that indicates an advantage for the observation of fractional QHE in BLG.

B. $\nu_n = 2/5$ and $\nu_n = 2/3$ states

According to the composite fermion theory³³, $\nu = m/(1+2m)$ fractional quantum Hall states are mapped onto the $\nu_{\text{eff}} = m$ integer quantum Hall states. In contrast to the case of $\nu = 1/3$, where the ground state is mapped onto the $n = 1$ pseudospin *polarized* integer quantum Hall state, $\nu = 2/5$ fractional quantum Hall state is mapped onto the $n = 2$ integer quantum Hall state, where the lowest LL is doubly occupied by pseudospin-up and pseudospin-down electrons. We therefore expect pseudospin unpolarized ground state at $\nu_n = 2/5$. Here we calculate the ground state pseudospin polarization of SLG and BLG at $\nu_n = 2/5$ to see whether this naive expectation is correct even in graphene.

The pseudospin-polarization in the ground state is studied by calculating the polarization energy $E_{P=1} - E_{P=0}$ which is the energy difference between the polarized $P = 1$ state and the unpolarized $P = 0$ state. This energy difference is plotted in Fig. 4 for the $n = 0$ and 1 LLs in SLG as a function of N_e . This figure clearly shows that the ground state pseudospin at $\nu_n = 2/5$ in the

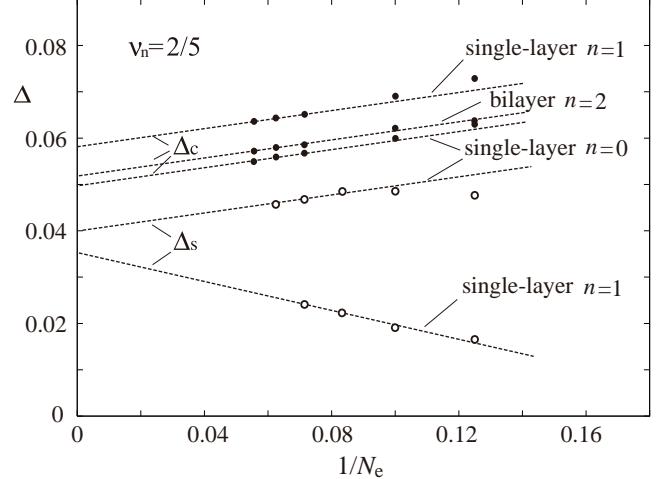


FIG. 5: The pseudospin (valley) polarized excitation gap Δ_c and the pseudospin (valley) unpolarized excitation gap Δ_s at $\nu_n = 2/5$ in $n = 0$ and 1 LLs of SLG and in the $n = 2$ LL of BLG.

$n = 0$ LL of SLG is unpolarized, because $E_{P=1} - E_{P=0}$ increases with the increase in the number of electrons N_e . This is consistent with the prediction of the composite fermion theory. On the other hand, in the $n = 1$ LL of SLG, $E_{P=1} - E_{P=0}$ decreases with the increase in N_e . This behavior suggests the polarized ground state.

To confirm the polarized ground state at $\nu_n = 2/5$ in the $n = 1$ LL of SLG, we slightly change the Haldane's pseudo potential V_2 , which acts only for electron pairs whose relative angular momentum m is 2. Since antisymmetrized wave function of polarized pseudospin state does not contain electron pairs with $m = 2$, only the energy of pseudospin polarized state is independent of V_2 . We therefore systematically control the energy difference between $E_{P=1}$ and $E_{P=0}$ by changing V_2 . The V_2 -dependence of $E_{P=1} - E_{P=0}$ is shown in the inset of Fig. 4 as a function of $\delta V_2 = V_2 - V_2^{\text{SLG}(n=1)}$ where $V_2^{\text{SLG}(n=1)}$ corresponds to the original V_2 in the $n = 1$ LL of SLG. This V_2 -dependence shows systematic change in the size dependence of $E_{P=1} - E_{P=0}$ at $\delta V_2 = -0.002$. Since δV_2 is negative at the transition, where $E_{P=1} - E_{P=0} = 0$ in the thermodynamic limit, the pseudospins in the ground state at $\nu_n = 2/5$ in $n = 1$ LL of SLG is fully polarized in large systems. Similar analysis on the size dependence of the polarization energy also shows that the pseudospins in the ground state of $n = 2$ LL of BLG are fully polarized at $\nu_n = 2/5$.

The elementally charge excitation energies at $\nu_n = 2/5$ are presented in Fig. 5. In the case of $n = 0$ LL of SLG, the pseudospin (valley) polarized excitation gap Δ_c is calculated supposing pseudospins in the ground state are fully polarized, although they are unpolarized in the true ground state. The extrapolated value of Δ_c in the $n = 0$ LL of SLG is then $0.05 e^2/(\epsilon \ell_B)$ in a good agreement with the previous work.³⁰ In the $n = 1$ LL of SLG, the

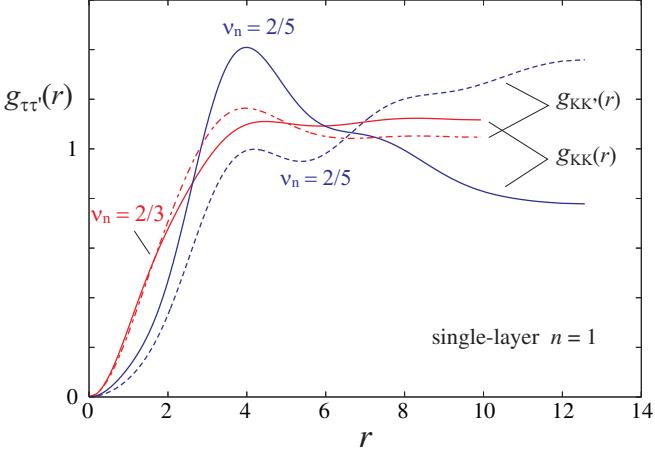


FIG. 6: (Color online) Two-particle correlation functions $g_{\tau\tau}(r)$ of quasi-hole skyrmion state at $\nu_n = 2/5$ and quasiparticle state at $\nu_n = 2/3$ in the $n = 1$ LL of SLG. r is in units of ℓ_B .

pseudospins in the ground state are fully polarized and Δ_c is 0.059. In BLG, $\Delta_c = 0.052$, which is also slightly larger than Δ_c in the $n = 0$ LL of SLG. The largest Δ_c at $\nu_n = 2/5$ is obtained in the $n = 1$ LL of SLG. This feature is the same as the case of $\nu_n = 1/3$ shown in Fig. 3.

The pseudospin (valley) unpolarized excitation gap Δ_s at $\nu_n = 2/5$ are also shown in Fig. 5. The Δ_s from the unpolarized ground state in the $n = 0$ LL of SLG is $0.04 e^2/(\epsilon\ell_B)$, and the Δ_s from the polarized ground state in the $n = 1$ LL of SLG $0.035 e^2/(\epsilon\ell_B)$. These Δ_s are smaller than Δ_c similarly to the case of $\nu_n = 1/3$. In the $n = 2$ LL of BLG, Δ_s is larger than Δ_c . The lowest gap in BLG is then given by Δ_c , which is $0.052 e^2/(\epsilon\ell_B)$.

In the $n = 1$ LL of SLG, the ground state is pseudospin polarized while the lowest charge excited state is pseudospin unpolarized. The pseudospin structure in the unpolarized excited states at $\nu_1 = 2/5$ is shown in Fig. 2 (b), which indicates elementally charge excitations are characterized by valley skyrmions. The two-particle correlation functions $g_{\tau\tau'}(r)$ ^{24,28,29} at $\nu_1 = 2/5$ are presented in Fig. 6, which show that the correlation function between the electrons in the same valley $g_{KK}(r)$ in the unpolarized quasi-hole state has peak structure around the origin while the correlation function between the electrons in different valleys $g_{KK'}(r)$ has the maximum at the opposite side on the sphere.^{15,24} These results indicate the elementally charge excitations at $\nu_n = 2/5$ in the $n = 1$ LL of SLG are valley-pseudospin textures similar to skyrmion excitations in the quantum Hall ferromagnetic states at $\nu_n = 1$ and $\nu_n = 1/3$, although their origin and detailed properties are not obvious.

We finally study the valley-polarization in the ground state and the elementally excitations at $\nu_n = 2/3$. Similarly to the case of the $\nu_n = 2/5$ fractional quantum Hall state, the $\nu_n = 2/3$ fractional quantum Hall state is mapped onto $n = 2$ integer quantum Hall state within

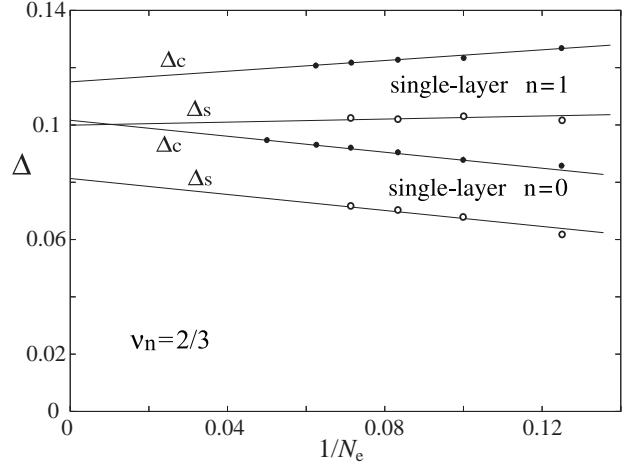


FIG. 7: The pseudospin (valley) polarized excitation gap Δ_c and the pseudospin (valley) unpolarized excitation gap Δ_s at $\nu_n = 2/3$ in $n = 0$ and 1 LLs of SLG.

a mean-field analysis of the composite fermion theory. Thus the pseudospin unpolarized ground state is expected. Indeed, the ground state at $\nu_n = 2/3$ in the $n = 0$ LL of SLG is pseudospin unpolarized. However, in the $n = 1$ LL of SLG and the $n = 2$ LL of BLG, the size dependence of the polarization energy shows the pseudospins are fully polarized in the ground state.

The charge excitation energies from the ground states at $\nu_n = 2/3$ are presented in Fig. 7. The polarized excitation gap Δ_c in the $n = 0$ LL of SLG is calculated from the energy difference between the fully polarized ground state and the fully polarized charge excited state, although the true ground state is the unpolarized state. The extrapolated value of Δ_c in the thermodynamic limit is then $0.101 e^2/(\epsilon\ell_B)$ in the $n = 0$ LL of SLG, which is the same as Δ_c at $\nu = 1/3$ because of the particle-hole symmetry of the single component quantum Hall system. In the $n = 1$ LL of SLG, the pseudospins in the ground state are polarized and Δ_c is 0.115 in the thermodynamic limit. In BLG, $\Delta_c = 0.103$, which is also the same as Δ_c at $\nu_n = 1/3$.

The valley unpolarized excitation gap Δ_s from the unpolarized true ground state in the $n = 0$ LL of SLG is $0.08 e^2/(\epsilon\ell_B)$. Similarly to the case of $\nu = 1/3$, Δ_s is smaller than Δ_c . In the $n = 1$ LL of SLG, we find Δ_s is 0.10 in the thermodynamic limit, which is also smaller than Δ_c . The valley unpolarized elementally charge excitations at $\nu_n = 2/3$ in the $n = 1$ LL of SLG are not skyrmion-like pseudospin-textured state, in contrast to the case of $\nu = 2/5$ in the $n = 1$ LL of SLG as shown in Fig. 6, where short-range correlation g_{KK} is smaller than $g_{KK'}$ for $\nu_n = 2/3$.

TABLE I: The pseudospin (valley) polarization in the ground state and the lowest charge excited state, and the unpolarized excitation gap Δ_s and the polarized excitation gap Δ_c extrapolated to the thermodynamic limit at the effective filling ν_n in the n th LL of single-layer graphene (SLG) and bilayer graphene (BLG).

ν_n	LL	ground state	excited state	Δ_s	Δ_c
1	SLG	0	polarized	skyrmion	0.63
1	SLG	1	polarized	skyrmion	0.28
1	SLG	2	polarized	skyrmion	0.42
1	BLG	2	polarized	skyrmion	0.48
1/3	SLG	0	polarized	skyrmion	0.05 0.101
1/3	SLG	1	polarized	skyrmion	0.03 0.115
1/3	BLG	2	polarized	polarized	0.103
2/5	SLG	0	unpolarized	unpolarized	0.04 (0.050) ³⁴
2/5	SLG	1	polarized	skyrmion	0.035 0.058
2/5	BLG	2	polarized	polarized	0.052
2/3	SLG	0	unpolarized	unpolarized	0.08 (0.101) ³⁴
2/3	SLG	1	polarized	unpolarized	0.10 0.115
2/3	BLG	2	polarized	polarized	0.103

IV. DISCUSSION

Our DMRG calculation confirms various types of quantum Hall states in graphene at $\nu_n = 1, 1/3, 2/5$ and $2/3$ in the $n = 0$ and 1 LLs of SLG and in the $n = 2$ LL of BLG. These results are summarized in Table I, where the pseudospin polarizations for the ground state and the

lowest charge excited state are listed with the elementally charge excitation energies Δ_c and Δ_s . The elementally charge excitations are obtained by increasing or decreasing the flux quantum number N_ϕ by 1 for $\nu = 1$ and $1/3, 2/5$, and by increasing or decreasing the flux quantum number with adding or removing single electron in the system for $\nu = 2/3$. We have studied both (a) pseudospin polarized excitations (Laughlin's quasiholes and quasiparticles) and (b) pseudospin unpolarized excitations (quasihole skyrmions and quasiparticle skyrmions at $\nu_n = 1$ and $1/3$ in the $n = 0$ and 1 LLs of SLG, and at $\nu_n = 1$ in the $n = 2$ LL of BLG).

The activation energies obtained in finite systems are extrapolated to the thermodynamic limit, which give theoretical predictions for future experimental studies of the fractional quantum Hall states in graphene. Our results show that the gaps ($\Delta_s \simeq 0.1e^2/\ell_B$) at $1/3$ and $2/3$ effective fillings in the excited LL in bilayer graphene are larger than those in conventional quantum well and single layer graphene. Therefore bilayer graphene is a good candidate for future experimental observation of the fractional quantum Hall effects.

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